$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$

The formula

$$\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2},$$

sometimes called a *Dirichlet integral* has drawn lots of attention. G.H. Hardy [2] has a note ranking various proofs of this formula. Later [3] he reconsidered his ranking and added a proof discovered by A. C. Dixon [1]. Hardy didn't know how to rank Dixon's proof using his criterion and assigned it a low ranking. I will assign it a high ranking. I like it a lot. This note will present Dixon's proof.

The convergence of $\int_0^\infty \frac{\sin x}{x}$ can be proved using the mean value theorem for integrals as stated in problem #7, section 4.2 of Folland (it's also true if ϕ is decreasing):

$$|\int_{B_1}^{B_2} \frac{\sin x}{x}| \le \frac{1}{B_1}|\int_{B_1}^{c} \sin x| + \frac{1}{B_2}|\int_{c}^{B_2} \sin x| \le 2(\frac{1}{B_1} + \frac{1}{B_2}) \to 0.$$

These are the steps that I will verify:

1. let

$$u_n = \int_0^{\frac{\pi}{2}} \sin 2nx \cot x \; dx, \tag{1}$$

$$v_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x} dx. \tag{2}$$

2. Using elementary arguments We will prove

   (a)

$$\lim_{n\to\infty} v_n = \int_0^\infty \frac{\sin x}{x}.$$

   (b)

$$u_n = \frac{\pi}{2}.$$

   (c)

$$\lim_{n\to\infty} (u_n - v_n) = 0.$$

Proof. (a) Let $t = 2nx$ and we see that $v_n = \int_0^{n\pi} \frac{\sin t}{t} dt \to \int_0^\infty \frac{\sin t}{t} dt$.

(b) We use some trigonometry. We will prove (b) by induction on n. We first notice

$$u_1 = \int_0^{\frac{\pi}{2}} \sin 2x \cot x \; dx = \int_0^{\frac{\pi}{2}} 2\cos^2 x dx = \int_0^{\frac{\pi}{2}} (\cos 2x + 1) dx = \frac{\pi}{2}.$$

The inductive step uses the following identities:

$$\sin(2n+2)x - \sin 2nx = 2\cos(2n+1)x\sin x, \tag{3}$$
$$2\cos(2n+1)x\cos x = \cos(2n+2)x + \cos(2nx), \tag{4}$$

which are proved using

$$\sin(2n+1\pm 1)x = \pm\cos(2n+1)x\sin x + \sin(2n+1)x\cos x, \tag{5}$$
$$\cos(2n+1\pm 1)x = \cos(2n+1)x\cos x \mp \sin(2n+1)x\sin x, \tag{6}$$

to show that

$$u_{n+1} = u_n = \frac{\pi}{2}.$$

(c) Finally we notice that $\frac{1}{x} - \cot x$ can be defined to be C^1 at $x = 0$ and hence we can integrate by parts

$$v_n - u_n = \int_0^{\frac{\pi}{2}} (\frac{1}{x} - \cot x)\sin 2nx dx = [-\frac{\cos 2nx}{2n}(\frac{1}{x} - \cot x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\cos 2nx}{2n}(\frac{1}{x^2} - \csc^2 x)dx.$$

Each of these terms goes to 0 as $n \to \infty$. This result also follows from the Riemann-Lebesgue lemma. □

I'll include here a proof in the spirit of Dixon's proof, although it is not his proof. I'll make use of

Theorem 1. *(Riemann-Lebesgue Lemma). Suppose f is Riemann integrable on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f(x)\sin nx dx = 0.$$

In this statement $\sin nx$ *can be replaced by* $\cos nx$ *or* $e^{\pm inx}$.

We next state a useful identity

Proposition 1.

$$1 + 2\cos 2y + 2\cos 4y + \cdots + 2\cos 2ny = \frac{\sin(2n+1)y}{\sin y}.$$

This follows from the following string of identities:

$$\begin{aligned} e^{-inx} + e^{-i(n-1)x} + \cdots + 1 + e^{ix} + \cdots + e^{inx} &= e^{-inx}\frac{e^{i(n+1/2)}(e^{i(n+1/2)x} - e^{-i(n+1/2)x})}{e^{ix/2}(e^{ix/2} - e^{-ix/2})} \\ &= \frac{\sin(n+1/2)x}{\sin x/2}. \end{aligned}$$

From the proposition it follows that

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)y}{\sin y} dy = \frac{\pi}{2}.$$

It is also true that

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)y}{y} dy = \int_0^{\frac{(2n+1)\pi}{2}} \frac{\sin t}{t} dt \to \int_0^{\infty} \frac{\sin t}{t} dt.$$

The Riemann-Lebesgue lemma implies

$$\int_0^{\frac{\pi}{2}} (\frac{1}{y} - \frac{1}{\sin y})\sin((2n+1)y)dy, \to 0.$$

and that concludes the proof.

References

[1] Dixon, A. C., Proof That $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$, *The Mathematical Gazette*, Vol. 6, No. 96 (Jan., 1912), pp. 223-224.

[2] Hardy, G. H. , The Integral $\int_0^\infty \frac{\sin\ x}{x} dx$, *The Mathematical Gazette*, Vol. 5, No. 80 (Jun. - Jul., 1909), pp. 98-103.

[3] Hardy, G. H., Further Remarks on the Integral $\int_0^\infty \frac{\sin x}{x} dx$, *The Mathematical Gazette*, Vol. 8, No. 124 (Jul., 1916), pp. 301-303.